ALGEBRA IN THE SUPEREXTENSIONS OF GROUPS, III: MINIMAL LEFT IDEALS OF $\lambda(\mathbb{Z})$

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Abstract. We prove that the minimal left ideals of the superextension $\lambda(\mathbb{Z})$ of the discrete group $\mathbb{Z}$ of integers are metrizable topological semigroups, topologically isomorphic to minimal left ideals of the superextension $\lambda(\mathbb{Z}_2)$ of the compact group $\mathbb{Z}_2$ of integer 2-adic numbers.

The superextension $\lambda(X)$ of a discrete group X is the compact Hausdorff right-topological semigroup consisting of maximal linked systems on X and endowed with the semigroup operation $\mathcal{A} * \mathcal{B} = \{A \subset X : \{x \in X : x^{-1}A \in \mathcal{B}\} \in \mathcal{A}\}$.

Introduction

After the topological proof (see [HS, p.102], [H2]) of Hindman theorem [H1], topological methods become a standard tool in the modern combinatorics of numbers, see [HS], [P]. The crucial point is that any semigroup operation $*$ defined on any discrete space X can be extended to a right-topological semigroup operation on $\beta(X)$, the Stone-Čech compactification of X. The extension of the operation from X to $\beta(X)$ can be defined by the simple formula:

$$\mathcal{U} * \mathcal{V} = \Big\{ \bigcup_{x \in U} x * V_x : U \in \mathcal{U},\ \{V_x\}_{x \in U} \subset \mathcal{V} \Big\}, \tag{1}$$

where $\mathcal{U}, \mathcal{V}$ are ultrafilters on X. Endowed with the so-extended operation, the Stone-Čech compactification $\beta(X)$ becomes a compact right-topological semigroup. The algebraic properties of this semigroup (for example, the existence of idempotents or minimal left ideals) have important consequences in combinatorics of numbers, see [HS], [P].

The Stone-Čech compactification $\beta(X)$ of X is the subspace of the double power-set $\mathcal{P}(\mathcal{P}(X))$, which is a complete lattice with respect to the operations of union and intersection. In [G_2] it was observed that the semigroup operation extends not only to $\beta(X)$ but also to the complete sublattice $G(X)$ of $\mathcal{P}(\mathcal{P}(X))$, generated by $\beta(X)$. This complete sublattice consists of all inclusion hyperspaces over X.

By definition, a family $\mathcal{F}$ of non-empty subsets of a discrete space X is called an *inclusion hyperspace* if $\mathcal{F}$ is monotone in the sense that a subset $A \subset X$ belongs to $\mathcal{F}$ provided A contains some set $B \in \mathcal{F}$. Besides the operations of union and intersection, the set $G(X)$ possesses an important transversality operation assigning to each inclusion hyperspace $\mathcal{F} \in G(X)$ the inclusion hyperspace

$$\mathcal{F}^{\perp} = \{A \subset X : \forall F \in \mathcal{F}\ (A \cap F \neq \emptyset)\}.$$

This operation is involutive in the sense that $(\mathcal{F}^{\perp})^{\perp} = \mathcal{F}$.

1991 *Mathematics Subject Classification.* 20M12, 54B99, 54H10.

It is known that the family $G(X)$ of inclusion hyperspaces on X is closed in the double power-set $\mathcal{P}(\mathcal{P}(X)) = \{0,1\}^{\mathcal{P}(X)}$ endowed with the natural product topology. The induced topology on $G(X)$ can be described directly: it is generated by the sub-base consisting of the sets

$$U^+ = \{\mathcal{F} \in G(X) : U \in \mathcal{F}\} \text{ and } U^- = \{\mathcal{F} \in G(X) : U \in \mathcal{F}^\perp\}$$

where U runs over subsets of X. Endowed with this topology, $G(X)$ becomes a Hausdorff supercompact space. The latter means that each cover of $G(X)$ by the sub-basic sets has a 2-element subcover.

The extension of a binary operation $*$ from X to $G(X)$ can be defined in the same way as for ultrafilters, i.e., by the formula (1) applied to any two inclusion hyperspaces $\mathcal{U}, \mathcal{V} \in G(X)$. In [G$_2$] it was shown that for an associative binary operation $*$ on X the space $G(X)$ endowed with the extended operation becomes a compact right-topological semigroup. Besides the Stone-Čech extension, the semigroup $G(X)$ contains many important spaces as closed subsemigroups. In particular, the space

$$\lambda(X) = \{\mathcal{F} \in G(X) : \mathcal{F} = \mathcal{F}^\perp\}$$

of maximal linked systems on X is a closed subsemigroup of $G(X)$. The space $\lambda(X)$ is well-known in General and Categorial Topology as the *superextension* of X, see [vM], [TZ]. Endowed with the extended binary operation, the superextension $\lambda(X)$ of a semigroup X is a supercompact right-topological semigroup containing $\beta(X)$ as a subsemigroup.

The thorough study of algebraic properties of the superextensions $\lambda(X)$ of groups X was started in [BGN] and continued in [BG$_2$]. In this paper we concentrate at describing the minimal (left) ideals of $\lambda(X)$.

Understanding the structure of minimal left ideals of the semigroup $\beta(X)$ had important combinatorial consequences. For example, properties of ultrafilters from a minimal left ideal of $\beta(X)$ were exploited in the topological proof of the classical Van der Waerden Theorem [HS, 14.3] due to Fustenberg and Katznelson [FK]. Minimal left ideals of the semigroup $\beta(\mathbb{Z})$ play also an important role in Topological Dynamics, see [BB], [BF], [HS, Ch.19]. We believe that studying the structure of minimal (left) ideals of the semigroups $\lambda(X)$ also will have some combinatorial or dynamical consequences.

The main result of this paper is Theorem 5.1 asserting that the minimal left ideals of the semigroup $\lambda(\mathbb{Z})$ are compact metrizable topological semigroups topologically isomorphic to minimal left ideals of the superextension $\lambda(\mathbb{Z}_2)$ of the (compact metrizable) group $\mathbb{Z}_2$ of integer 2-adic numbers.

1. Right-topological semigroups

In this section we recall some information from [HS] related to right-topological semigroups. By definition, a right-topological semigroup is a topological space S endowed with a semigroup operation $* : S \times S \to S$ such that for every $a \in S$ the right shift $r_a : S \to S$, $r_a : x \mapsto x * a$, is continuous. If the semigroup operation $* : S \times S \to S$ is continuous, then $(S, *)$ is a *topological semigroup.*

A non-empty subset I of a semigroup S is called a *left* (resp. *right*) *ideal* if $SI \subset I$ (resp. $IS \subset I$). If I is both a left and right ideal in S, then I is called an *ideal* in S. Observe that for every $x \in S$ the set $Sx = \{sx : s \in S\}$ (resp. $xS = \{xs : s \in S\}$) is a left (resp. right) ideal in S. Such an ideal is called *principal.* An ideal $I \subset S$

is called *minimal* if any ideal of S that lies in I coincides with I. By analogy we define minimal left and right ideals of S. It is easy to see that each minimal left (resp. right) ideal I is principal. Moreover, $I = Sx$ (resp. $I = xS$) for each $x \in I$.

If S is a compact Hausdorff right-topological semigroup, then each minimal left ideal in S, being principal, is closed in S. By [HS, 2.6], each left ideal in S contains a minimal left ideal. The union of all minimal left ideals of S coincides with the minimal ideal $K(S)$ of S, [HS, 2.8]. By [HS, 2.11], all the minimal left ideals of S are mutually homeomorphic.

An element z of a semigroup S is called a *right zero* in S if $xz = z$ for all $x \in S$. It is clear that $z \in S$ is a right zero in S if and only if the singleton $\{z\}$ is a (minimal) left ideal in S.

In the sequel we shall often use the following

Lemma 1.1. *Let X, Y be compact right-topological semigroups. If a semigroup homomorphism $h : X \to Y$ is injective on some minimal left ideal of X, then h is injective on each minimal left ideal of X.*

Proof. Assume that h is injective on a minimal left ideal Xa of X and take any other minimal left ideal Xb of X. By [HS, 2.11], the right shift $r_a : X \to X$, $r_a : x \mapsto xa$, is injective on Xb. Next, consider the right shift $r_{h(a)} : Y \to Y$, $r_{h(a)} : y \mapsto y \cdot h(a)$. It follows from the equality $h \circ r_a = r_{h(a)} \circ h$ and the injectivity of the maps $r_a|Xb$ and $h|Xa$ that the map $h|Xb$ is injective. □

2. Inclusion hyperspaces and superextensions

A family $\mathcal{L}$ of subsets of a set X is called *a linked system on* X if $A \cap B \neq \emptyset$ for all $A, B \in \mathcal{L}$. Such a linked system $\mathcal{L}$ is *maximal linked* if $\mathcal{L}$ coincides with any linked system $\mathcal{L}'$ on X that contains $\mathcal{L}$. Each (ultra)filter on X is a (maximal) linked system. A linked system $\mathcal{L}$ on X is maximal linked if and only if for any partition $X = A \cup B$ either A or B belongs to $\mathcal{L}$.

By $\lambda(X)$ we denote the family of all maximal linked systems on X. Since each ultrafilter on X is a maximal linked system, $\lambda(X)$ contain the Stone-Čech extension $\beta(X)$ of X. It is easy to see that each maximal linked system on X is an inclusion hyperspace on X and hence $\lambda(X) \subset G(X)$. Moreover, it can be shown that $\lambda(X) = \{\mathcal{A} \in G(X) : \mathcal{A} = \mathcal{A}^\perp\}$. Let also $N_2(X) = \{\mathcal{A} \in G(X) : \mathcal{A} \subset \mathcal{A}^\perp\}$ denote the family of all linked inclusion hyperspaces on X. By [G_1] both the subspaces $\lambda(X)$ and $N_2(X)$ are closed in the compact Hausdorff space $G(X)$.

Each function $f : X \to Y$ between sets X, Y induces a continuous map $Gf : G(X) \to G(Y)$ assigning to an inclusion hyperspace $\mathcal{A} \in G(X)$ the inclusion hyperspace

$$Gf(\mathcal{A}) = \{B \subset Y : f^{-1}(B) \in \mathcal{A}\} \in G(Y).$$

The function Gf maps $\lambda(X)$ into $\lambda(Y)$, so we can put $\lambda f = Gf|\lambda(X)$.

Given any semigroup operation $* : X \times X \to X$ on a set X we can extend this operation to $G(X)$ letting

$$\mathcal{U} * \mathcal{V} = \Big\{ \bigcup_{x \in U} x * V_x : U \in \mathcal{U},\ \{V_x\}_{x \in U} \subset \mathcal{V} \Big\}$$

for inclusion hyperspaces $\mathcal{U}, \mathcal{V} \in G(X)$. Equivalently, the product $\mathcal{U} * \mathcal{V}$ can be defined as

$$\mathcal{U} * \mathcal{V} = \{A \subset X : \{x \in X : x^{-1}A \in \mathcal{V}\} \in \mathcal{U}\}$$

where $x^{-1}A = \{z \in X : x * z \in A\}$. By [G$_2$] the so-extended operation turns $G(X)$ into a right-topological semigroup. The structure of this semigroup was studied in details in [G$_2$]. In particular, it was shown that for each group X the minimal left ideals of $G(X)$ are singletons containing *invariant* inclusion hyperspaces.

We call an inclusion hyperspace $\mathcal{A} \in G(X)$ *invariant* if $x\mathcal{A} = \mathcal{A}$ for all $x \in X$. More generally, given a subgroup $H \subset X$ we define $\mathcal{A}$ to be H*-invariant* if $x\mathcal{A} = \mathcal{A}$ for all $x \in H$.

It follows from the definition of the topology on $G(X)$ that the set $\overset{\leftrightarrow}{G}(X)$ of invariant inclusin hyperspaces is closed in $G(X)$ and coincides with the minimal ideal $K(G(X))$ of the semigroup $G(X)$. Consequently, $K(G(X))$ is a closed rectangular subsemigroup of $G(X)$. The *rectangularity* of $K(G(X))$ means that $\mathcal{A} \circ \mathcal{B} = \mathcal{B}$ for all $\mathcal{A}, \mathcal{B} \in K(G(X))$.

3. The minimal ideal of $\lambda(G)$ for odd groups

In this section we characterize groups G whose superextension $\lambda(G)$ has one-point minimal left ideals.

Following [BGN], we define a group G to be *odd* if the order of each element x of G is odd. If G is a finite odd group, then the maximal linked system

$$\mathcal{L} = \{A \subset G : |A| > |G|/2\}$$

is invariant. In fact, a group G possesses an invariant maximal linked system if and only if G is odd, see Theorem 3.2 of [BGN]. By Proposition 3.1 of [BGN], a maximal linked system $\mathcal{Z} \in \lambda(G)$ on a group G is invariant if and only if $\mathcal{Z}$ is a right zero of the semigroup $\lambda(G)$ if and only if the singleton $\{\mathcal{Z}\}$ is a minimal left ideal in $\lambda(G)$. Taking into account that the invariant maximal linked systems form a closed rectangular subsemigroup of $\lambda(G)$, we obtain the main result of this section.

Theorem 3.1. *A group G is odd if and only if all the minimal left ideals of $\lambda(G)$ are singletons. In this case the minimal ideal $K(\lambda(G))$ of $\lambda(G)$ is a closed rectangular semigroup consisting of invariant maximal linked systems.*

Given a subgroup H of a group G let $G/H = \{xH : x \in G\}$ and $\pi : G \to G/H$ denote the quotient map. It induces a continuous map $\lambda\pi : \lambda(G) \to \lambda(G/H)$ between the corresponding superextensions.

Lemma 3.2. *For any H-invariant maximal linked system $\mathcal{A} \in \lambda(H) \subset \lambda(G)$ the restriction of $\lambda\pi : \lambda(G) \to \lambda(G/H)$ to the principal left ideal $\lambda(G) * \mathcal{A}$ is injective.*

Proof. Fix a section $s : G/H \to G$ of π. For every $\mathcal{L} \in \lambda(G)$ let $\widetilde{\mathcal{L}} = \lambda\pi(\mathcal{L}) \in \lambda(G/H)$ be the projection of $\mathcal{L}$ onto G/H and $\mathcal{M} = \lambda s(\widetilde{\mathcal{L}}) \in \lambda(G)$ be the lift of $\widetilde{\mathcal{L}}$ by the section s.

We claim that $\mathcal{L}*\mathcal{A} = \mathcal{M}*\mathcal{A}$. Since $\mathcal{L}*\mathcal{A}$ and $\mathcal{M}*\mathcal{A}$ are maximal linked systems, it suffices to check that $\mathcal{L}*\mathcal{A} \subset \mathcal{M}*\mathcal{A}$. Take any set $\bigcup_{x\in L} x*A_x \in \mathcal{L}*\mathcal{A}$ where $L \in \mathcal{L}$ and $\{A_x\}_{x\in L} \subset \mathcal{A}$. Consider the set $M = s \circ \pi(L) \in \mathcal{M}$. For every point $y \in M$ find a point $x_y \in L$ with $y = s\pi(x_y)$ and observe that $yH = \pi(y) = \pi(x_y) = x_yH$, which implies $y^{-1}x_y \in H$ and hence $y^{-1}x_yA_{x_y} \in \mathcal{A}$ by the H-invariantness of $\mathcal{A}$. Since

$$\mathcal{M} * \mathcal{A} \ni \bigcup_{y\in M} y(y^{-1}x_y * A_{x_y}) = \bigcup_{y\in M} x_y * A_{x_y} \subset \bigcup_{x\in L} x * A_x$$

we conclude that $\bigcup_{x\in L} x * A_x \in \mathcal{M} * \mathcal{A}$.

Now we are able to prove that $\lambda\pi : \lambda(G) * \mathcal{A} \to \lambda(G/H)$ is injective. Take any two distinct elements $\mathcal{L}_1 * \mathcal{A} \neq \mathcal{L}_2 * \mathcal{A}$ of $\lambda(G) * \mathcal{A}$. For every $i \in \{1, 2\}$ consider the maximal linked systems $\widetilde{\mathcal{L}}_i = \lambda\pi(\mathcal{L}_i) = \lambda\pi(\mathcal{L}_i * \mathcal{A})$ and $\mathcal{M}_i = \lambda s(\widetilde{\mathcal{L}}_i)$. It follows from $\mathcal{M}_1 * \mathcal{A} = \mathcal{L}_1 * \mathcal{A} \neq \mathcal{L}_2 * \mathcal{A} = \mathcal{M}_2 * \mathcal{A}$ that $\mathcal{M}_1 \neq \mathcal{M}_2$ and hence

$$\lambda\pi(\mathcal{L}_1 * \mathcal{A}) = \widetilde{\mathcal{L}}_1 \neq \widetilde{\mathcal{L}}_2 = \lambda\pi(\mathcal{L}_2 * \mathcal{A}).$$

□

Corollary 3.3. *For a normal odd subgroup H of a group G the map $\lambda\pi : \lambda(G) \to \lambda(G/H)$ is injective on each minimal left ideal of $\lambda(G)$. Consequently, every minimal left ideal of $\lambda(G)$ is topologically isomorphic to a minimal left ideal of $\lambda(G/H)$.*

Proof. By Lemma 1.1, it suffices to show that $\lambda\pi$ is injective on some minimal left ideal. The group H, being odd, admits an H-invariant maximal linked system $\mathcal{A} \in \lambda(H) \subset \lambda(G)$. By Lemma 3.2 the homomorphism $\lambda\pi$ is injective on the left ideal $\lambda(G) * \mathcal{A}$ and hence is injective on any minimal left ideal contained in $\lambda(G) * \mathcal{A}$ (it exists because $\lambda(G)$ is a compact right-topological semigroup). □

4. Maximal invariant linked systems on groups

As we have seen in the preceding section, the property of a maximal system $\mathcal{L} \in \lambda(G)$ to be invariant is very strong and forces $\mathcal{L}$ to be a right zero of $\lambda(G)$. Such maximal linked systems exist only on odd groups.

On the other hand, maximal invariant linked systems exist on each group. An invariant linked inclusion hyperspace $\mathcal{L} \in \overset{\leftrightarrow}{N}_2(G)$ is called a *maximal invariant linked system* if $\mathcal{L} = \mathcal{L}'$ for any invariant linked inclusion hyperspace $\mathcal{L}' \in \overset{\leftrightarrow}{N}_2(G)$ enlarging $\mathcal{L}$. By the Zorn Lemma, each invariant linked inclusion hyperspace can be enlarged to a maximal invariant linked system.

Proposition 4.1. *For any maximal invariant linked system $\mathcal{L}_0$ on a group G the set*

$$\uparrow\!\mathcal{L}_0 = \{\mathcal{L} \in \lambda(G) : \mathcal{L} \supset \mathcal{L}_0\}$$

is a left ideal in $\lambda(G)$.

Proof. Let $\mathcal{A}, \mathcal{B} \in \lambda(X)$ be maximal linked systems with $\mathcal{L}_0 \subset \mathcal{B}$. Then for every subset $L \in \mathcal{L}_0$ we get

$$L = \bigcup_{x\in G} x(x^{-1}L) \in \mathcal{A} * \mathcal{L}_0 \subset \mathcal{A} * \mathcal{B}$$

which means that $\mathcal{L}_0 \subset \mathcal{A} * \mathcal{B}$. □

Observe that $\mathcal{L}_0 \subset \mathcal{L} \subset \mathcal{L}_0^{\perp}$ for every $\mathcal{L} \in \uparrow\!\mathcal{L}_0$. The following theorem shows that the difference $\mathcal{L}_0^{\perp} \setminus \mathcal{L}_0$ (and consequently, $\mathcal{L} \setminus \mathcal{L}_0$) is relatively small (for the group $G = \mathbb{Z}$ it is countable!).

Theorem 4.2. *If $\mathcal{L}_0$ is a maximal invariant linked system on an Abelian group G, then for any subset $A \in \mathcal{L}_0^{\perp} \setminus \mathcal{L}_0$ there is a point $x \in G$ such that $xA = G \setminus A$ and consequently, $A = x^2A$.*

Proof. Fix a subset $A \in \mathcal{L}_0^\perp \setminus \mathcal{L}_0$. We claim that

$$aA \cap A = \emptyset \tag{2}$$

for some $a \in G$. Assuming the converse, we would conclude that the family $\{xA : x \in G\}$ is linked and then the invariant linked system $\mathcal{L}_0 \cup \{xA : x \in G\}$ is strictly larger than $\mathcal{L}_0$, which impossible because of the maximality of $\mathcal{L}_0$.

Next, we find $b \in G$ with

$$A \cup bA = G. \tag{3}$$

Assuming that no such a point b exist, we conclude that for any $x, y \in G$ the union $xA \cup yA \neq G$. Then $(G \setminus xA) \cap (G \setminus yA) = G \setminus (xA \cup yA) \neq \emptyset$, which means that the family $\{G \setminus xA : x \in G\}$ is linked and invariant. We claim that $G \setminus A \in \mathcal{L}_0^\perp$. Assuming the converse, we would conclude that $G \setminus A$ misses some set $L \in \mathcal{L}_0$. Then $L \subset A$ and hence $A \in \mathcal{L}_0$ which is not the case. Thus $G \setminus A \in \mathcal{L}_0^\perp$ and hence $\{G \setminus xA : x \in G\}$ because $\mathcal{L}_0^\perp$ is invariant. Since $\mathcal{L}_0 \cup \{G \setminus xA : x \in G\}$ is an invariant linked system containing $\mathcal{L}_0$, the maximality of $\mathcal{L}_0$ guarantees that $G \setminus A \in \mathcal{L}_0$ which contradicts $A \in \mathcal{L}_0^\perp$.

Finally we show that $G \setminus A = aA = bA$. Observe that (2) and (3) imply that $aA \subset bA$ and hence $A \subset a^{-1}bA$. On the other hand, (2) and (3) are equivalent to $a^{-1}A \cap A = \emptyset$ and $b^{-1}A \cup A = G$, which implies $a^{-1}A \subset b^{-1}A$ and this yields $ba^{-1}A \subset A$. Unifying this inclusion with $A \subset a^{-1}bA = ba^{-1}A$, we conclude that $ba^{-1}A = A$ and hence $bA = aA$. Now looking at (2) and (3) we see that $G \setminus A = aA = bA$. □

5. Minimal left ideals of $\lambda(\mathbb{Z})$

In this section we apply the results of the preceding sections to describe the structure of minimal left ideals of the semigroup $\lambda(\mathbb{Z})$. It turns out that they are isomorphic to minimal left ideals of the superextension $\lambda(\mathbb{Z}_2)$ of the compact topological group $\mathbb{Z}_2$ of integer 2-adic numbers. We recall that $\mathbb{Z}_2 = \varprojlim C_{2^k}$ is a totally disconnected compact metrizable Abelian group, which is the limit of the inverse sequence

$$\cdots \to C_{2^n} \to \cdots \to C_8 \to C_4 \to C_2$$

of cyclic 2-groups C_{2^n}. Let $\pi : \mathbb{Z} \to \mathbb{Z}_2$ denote the canonic (injective) homomorphism of $\mathbb{Z}$ into $\mathbb{Z}_2$ (induced by the quotient maps $\pi_{2^k} : \mathbb{Z} \to \mathbb{Z}/2^k\mathbb{Z} = C_{2^k}$, $k \in \mathbb{N}$).

By the continuity of the functor λ in the category of compact Hausdorff spaces (see [TZ, 2.3.2]), the superextension $\lambda(\mathbb{Z}_2)$ can be identified with the limit of the inverse sequence

$$\cdots \to \lambda(C_{2^n}) \to \cdots \to \lambda(C_8) \to \lambda(C_4) \to \lambda(C_2)$$

of finite semigroups $\lambda(C_{2^k})$. This implies that $\lambda(\mathbb{Z}_2)$ is a metrizable zero-dimensional compact topological semigroup.

Theorem 5.1. *The homomorphism $\lambda\pi : \lambda(\mathbb{Z}) \to \lambda(\mathbb{Z}_2)$ is injective on each minimal left ideal of $\lambda(\mathbb{Z})$. Consequently, the minimal left ideals of the semigroup $\lambda(\mathbb{Z})$ are compact metrizable topological semigroups.*

Proof. By Lemma 1.1, it suffices to check that the homomorphism $\lambda\pi$ is injective on some minimal left ideal of $\lambda(\mathbb{Z})$. Fix any maximal invariant linked system $\mathcal{L}_0$ on $\mathbb{Z}$ (such a system exists by Zorn Lemma). By Proposition 4.1 the set $\uparrow\mathcal{L}_0 = \{\mathcal{L} \in \lambda(\mathbb{Z}) : \mathcal{L} \supset \mathcal{L}_0\}$ is a left ideal which necessarily contains a minimal left ideal

I of $\lambda(\mathbb{Z})$. We claim that the homomorphism $\lambda\pi : \lambda(\mathbb{Z}) \to \lambda(\mathbb{Z}_2)$ is injective on I. Given two different maximal linked system $\mathcal{A}, \mathcal{B} \in I$ we need to check that $\lambda\pi(\mathcal{A}) \neq \lambda\pi(\mathcal{B})$.

Since the superextension $\lambda(\mathbb{Z}_2)$ is the limit of the inverse sequence

$$\cdots \to \lambda(C_{2^n}) \to \cdots \to \lambda(C_8) \to \lambda(C_4) \to \lambda(C_2),$$

the inequality $\lambda\pi(\mathcal{A}) \neq \lambda\pi(\mathcal{B})$ will follow as soon as we find $k \in \mathbb{N}$ such that $\lambda\pi_{2^k}(\mathcal{A}) \neq \lambda\pi_{2^k}(\mathcal{B})$ where $\lambda\pi_{2^k} : \lambda(\mathbb{Z}) \to \lambda(C_{2^k})$ is the homomorphism induced by the quotient homomorphism $\pi_{2^k} : \mathbb{Z} \to C_{2^k}$.

Pick any set $A \in \mathcal{A} \setminus \mathcal{B}$. Since $A \in \mathcal{L}_0^\perp \setminus \mathcal{L}_0$, we can apply Theorem 4.2 to conclude that $A = 2n + A$ for some positive number $n \in \mathbb{Z}$. The later equality means that $A = \pi_{2n}^{-1}(\pi_{2n}(A))$ is the complete preimage of the set $\pi_{2n}(A)$ under the quotient homomorphism $\pi_{2n} : \mathbb{Z} \to \mathbb{Z}/2n\mathbb{Z} = C_{2n}$. It follows that $\pi_{2n}(A) \in \lambda\pi_{2n}(\mathcal{A}) \setminus \lambda\pi_{2n}(\mathcal{B})$ and hence $\lambda\pi_{2n}(\mathcal{A}) \neq \lambda\pi_{2n}(\mathcal{B})$.

Write the number $2n$ as the product $2n = 2^k \cdot m$ for some odd number m and find a (unique) subgroup $H \subset C_{2n}$ of order $|H| = m$. It follows that the quotient group C_{2n}/H can be identified with the cyclic 2-group C_{2^k} so that $q \circ \pi_{2n} = \pi_{2^k}$ where $q : C_{2n} \to C_{2n}/H = C_{2^k}$ is the quotient homomorphism. Corollary 3.3 guarantees that the homomorphism $\lambda q : \lambda(C_{2n}) \to \lambda(C_{2^k})$ is injective on each minimal left ideal of $\lambda(C_{2n})$. In particular, it is injective on the minimal left ideal $\lambda\pi_{2n}(I)$. Consequently, $\lambda\pi_{2^k}(\mathcal{A}) = \lambda q(\tilde{\mathcal{A}}) \neq \lambda q(\tilde{\mathcal{B}}) = \lambda\pi_{2^k}(\mathcal{B})$. This completes the proof of the injectivity of $\lambda\pi : \lambda(\mathbb{Z}) \to \lambda(\mathbb{Z}_2)$ on the left ideal I and consequently, on each minimal left ideal J of $\lambda(\mathbb{Z})$.

Since minimal left ideals of $\lambda(\mathbb{Z})$ are compact, the restriction $\lambda\pi|J$ is a topological isomorphism of J onto the minimal left ideal $\lambda\pi(J)$ of $\lambda(\mathbb{Z}_2)$. Since $\lambda(\mathbb{Z}_2)$ is a metrizable topological semigroup, so are the semigroups $\lambda\pi(J)$ and J. □

6. Some Open Problems

We saw in Theorem 3.1 that the minimal ideal $K(\lambda(G))$ of the superextension of an odd group G is a compact topological semigroup.

Problem 6.1. *Characterize groups G such that the minimal ideal $K(\lambda(G))$ is closed in $\lambda(G)$. Is the minimal ideal $K(\lambda(\mathbb{Z}))$ closed in $\lambda(\mathbb{Z})$? Is $K(\lambda(\mathbb{Z}))$ a topological semigroup?*

Problem 6.2. *Characterize groups G such that the minimal left ideals of $\lambda(G)$ are (metrizable) topological semigroups.*

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